École Normale Supérieure Paris
International Selection 2015
Major Mathematics

Exercise 1

Let $d, n \geq 1$ be integers, and let $f_1, \dots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ be diffeomorphisms such that there exists a constant $0 < c < 1$ with the property that

$$|||Df_i(x)||| \leq c$$

for every $x \in \mathbb{R}^d$ and every $i = 1, \dots, n$. Here, $Df_i(x)$ stands for the derivative of f_i at x, and $|||.|||$ denotes the operator norm associated to the euclidean norm: namely for any $M \in \mathrm{End}(\mathbb{R}^n)$, one has

$$|||M||| = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{||Mx||}{||x||}, \text{ where } ||(x_1, \dots, x_n)|| = \sqrt{\sum_i x_i^2}.$$

The goal of the exercise is to prove that there exists a unique non-empty compact subset $K \subset \mathbb{R}^d$ satisfying

$$K = \bigcup_{i=1,\dots,n} f_i(K) \tag{1}$$

and to study this set.

1. Prove that when $n = 1$ there is a unique non-empty compact set K satisfying (1). What is the nature of K in this case?

2. We introduce the set $\mathcal{K}$ of non-empty compact subsets of $\mathbb{R}^d$, and we equip it with the Hausdorff distance, defined as

$$d(K_1, K_2) := \inf\{\varepsilon > 0 \mid K_1 \subset K_2^\varepsilon \text{ and } K_2 \subset K_1^\varepsilon\},$$

where $K^\varepsilon := \{x \in \mathbb{R}^d \mid d(x, K) \leq \varepsilon\}$. Prove that $(\mathcal{K}, d)$ is a complete metric space.

3. Prove that there exists a unique non-empty compact set K satisfying (1).

4. Let $n = d = 2$, and let $f_1, f_2 : \mathbb{C} \simeq \mathbb{R}^2 \to \mathbb{C} \simeq \mathbb{R}^2$ be complex affine maps of the form

$$f_1(z) = \lambda z + 1 \text{ and } f_2(z) = \lambda z - 1 \text{ for } z \in \mathbb{C},$$

where $\lambda \in \mathbb{C}$ has absolute value $0 < |\lambda| < 1$. Show that if $|\lambda| < 1/2$, then K is not connected, and that its connected components are points.

5. (Bousch's theorem) Show that, *a contrario*, if $|\lambda| \geq \frac{1}{\sqrt{2}}$, then K is connected.

Hint: *To prove Bousch's theorem, you may first show that under the assumption $|\lambda| \geq \frac{1}{\sqrt{2}}$, one has $f_1(K) \cap f_2(K) \neq \emptyset$.*

Exercise 2

Let n be a positive integer number, and let E be a complex vector space of dimension n. Denote by $\mathcal{L}(E)$ the vector space of endomorphisms of E, and denote by E^* the dual space of E. The vector space of endomorphisms of E^* is denoted by $\mathcal{L}(E^*)$.

Consider a vector subspace $A \subset \mathcal{L}(E)$ such that A contains the identity endomorphism and is closed with respect to the composition. Put $A^* = \{a^* \mid a \in A\} \subset \mathcal{L}(E^*)$, where a^* denotes the transpose of a.

For any vector $v \in E$, denote by $A(v)$ the set $\{a(v) \mid a \in A\} \subset E$, and for any vector $u \in E^*$, denote by $A^*(u)$ the set $\{a^*(u) \mid a^* \in A^*\} \subset E^*$.

Assume that there is no proper subspace $\{0\} \subsetneq F \subsetneq E$ which is A-invariant, that is, such that $a(F) \subset F$ for each $a \in A$.

1. Let $v \in E$ and $u \in E^*$ be non-zero vectors. Show that $A(v) = E$ and $A^*(u) = E^*$.

2. Let v_1 and v_2 be two linearly independent vectors in E. Prove that there exists an endomorphism $a \in A$ such that $a(v_1) \neq 0$ and $a(v_2) = 0$.
Hint: *Show that, otherwise, there would exist an endomorphism $t \in \mathcal{L}(E)$ such that $t(v_2) = v_1$ and $(t \circ b)(v_2) = (b \circ t)(v_2)$ for any b in A, and obtain a contradiction.*

3. Prove that A contains an endomorphism of rank 1.

4. Prove that A contains all endomorphisms of rank 1 in $\mathcal{L}(E)$. Deduce that A coincides with $\mathcal{L}(E)$.

5. (Burnside's theorem) Let r be a positive integer number. Prove that there exists an integer number N (depending on n and r) such that any subgroup $G \subset GL(E)$ (where $GL(E)$ denotes the general linear group of E) whose elements have order at most r satisfies the following property: G is finite and its cardinality is at most N.

Exercise 3

The size of a finite set S is denoted by $|S|$.

For a finite set S, denote by $\mathbb{R}^S$ the $\mathbb{R}$-linear space of functions $g : S \to \mathbb{R}$. For a function $g \in \mathbb{R}^S$, the average value of g on S is denoted by $\mathbb{E}(g) := \frac{1}{|S|} \sum_{s \in S} g(s)$.

The vector space $\mathbb{R}^S$ has a natural scalar product given by $(g, h) := \mathbb{E}(g.h) = \frac{1}{|S|} \sum_{s \in S} g(s)h(s)$, and $||g||^2 := (g, g)$

For a subset $T \subset S$, the characteristic function of T is denoted by $\mathbf{1}_T \in \mathbb{R}^S$: it takes value 1 at each $t \in T$ and zero outside. The line in $\mathbb{R}^S$ spanned by $\mathbf{1}_T$ is denoted by L_T.

For a subset $T \subset S$ and $g \in \mathbb{R}^S$, the restriction of g to T is denoted by $g_{|T} \in \mathbb{R}^T$.

Recall that a partition $P = \{V_1, \ldots, V_m\}$ of a set V is a finite collection $V_1, \ldots, V_m$ of disjoint subsets of V with $\bigcup_{i=1}^m V_i = V$. Another partition $Q = \{U_1, \ldots, U_k\}$ of V is a refinement of P if for any $U_j \in Q$, there exists $V_i \in P$ with $U_j \subseteq V_i$.

*

1. For finite sets $T \subseteq S$, show that the orthogonal projection of a function $g \in \mathbb{R}^S$ to L_T is given by $\mathbb{E}(g_{|T})\mathbf{1}_T$.
2. Show that a partition P of S gives a set of pairwise orthogonal vectors $\mathbf{1}_X$, for $X \in P$.

**

From now on we will assume $S = V \times V$ for a finite set V. A partition P of V gives rise to a partition $P^2 = \{X \times Y \mid X, Y \in P\}$ of S. Define the subspace L_P of $\mathbb{R}^S$ by

$$L_P := \bigoplus_{X,Y \in P} L_{X \times Y},$$

and denote by π_P the orthogonal projection of $\mathbb{R}^S$ to L_P.

3. Show that if a partition Q is a refinement of a partition P of V, then $\pi_P \circ \pi_Q = \pi_P$. In particular, for any $g \in \mathbb{R}^S$, we have $||\pi_Q(g)||^2 = ||\pi_Q(g) - \pi_P(g)||^2 + ||\pi_P(g)||^2$.

Let now $f : S \to \{0,1\}$ be a fixed Boolean function. For two subsets $X, Y \subset V$, define

$$\mu_{X,Y} := \mathbb{E}(f_{|X\times Y}) = \frac{1}{|X||Y|} \sum_{x\in X, y\in Y} f(x,y).$$

Let $\epsilon > 0$ be a positive real number. The pair (X,Y) is called ϵ-regular with respect to f if for any subsets $A \subset X$ and $B \subset Y$ with $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$, we have $|\mu_{A,B} - \mu_{X,Y}| \leq \epsilon$; otherwise, (X,Y) is called ϵ-irregular.

4. Let $X, Y \subset V$. Define $g : X \times Y \to [-1,1]$ by $g := f_{|X\times Y} - \mu_{X,Y}$. Prove that if (X,Y) is ϵ-regular for f, then for any two functions $\alpha : X \to [0,1]$ and $\beta : Y \to [0,1]$, one has $|\,\mathbb{E}(g\alpha\beta)\,| \leq \epsilon$. Here $g\alpha\beta \in \mathbb{R}^{X\times Y}$ takes value $g(x,y)\alpha(x)\beta(y)$ at $(x,y) \in X \times Y$.

Hint: *Treat first the case of $\alpha = \mathbf{1}_A$ and $\beta = \mathbf{1}_B$ for subsets $A \subseteq X$ and $B \subseteq Y$.*

***

For a partition P of V, define the irregularity function $\mathrm{Irr}_P : S \to \{0,1\}$ by $\mathrm{Irr}_P(x,y) = 1$ if and only if $(x,y) \in X \times Y$ for an ϵ-irregular pair (X,Y), $X, Y \in P$.

A partition P is called ϵ-regular if $||\mathrm{Irr}_P||^2 = \mathbb{E}(\mathrm{Irr}_P) \leq \epsilon$, in other words, if

$$\sum_{\substack{X,Y\in P \\ (X,Y)\ \epsilon-\text{irregular}}} |X||Y| \leq \epsilon|V|^2.$$

5. Let X be a finite set and $M_1, \ldots, M_k$ a family of subsets of X. Show that there exists a partition Q_X of X of size at most 2^k such that each M_i is a disjoint union of elements of Q_X.

6. Prove that any ϵ-irregular partition of V has a refinement Q with $|Q| \leq |P|4^{|P|}$ and $||\pi_Q(f)||^2 \geq ||\pi_P(f)||^2 + \epsilon^5$.

****

Let $\epsilon > 0$ be a fixed positive real number. A partition $P = \{X_1, \ldots, X_m\}$ of V is called ϵ-balanced if there exists a subset $I \subset \{1, \ldots, m\}$ which verifies

- for all $i, j \in I$, we have $|X_i| = |X_j|$, and
- $|\bigcup_{i\notin I} X_i\,| \leq \epsilon|V|$.

7. Prove that any partition P of V contains a refinement Q such that Q is ϵ-balanced and $|Q| \leq (1 + \epsilon^{-1})|P|$.

*****

8. (Szemerédi regularity lemma) Show that a finite number of iterations of 6. and 7. proves the following result: For any $\epsilon > 0$, there exists N_ϵ such that for any finite set V and any Boolean function f on $S = V \times V$, there exists an ϵ-balanced ϵ-regular partition P of V of size at most N_ϵ.

******

Let $P = \{X_1, \ldots, X_m\}$ be an ϵ-regular ϵ-balanced partition of V for $f : S \to \{0,1\}$, with $m \leq N_\epsilon$ (whose existence is a consequence of question 8.), and let $k \leq m$ with $|X_1| = \cdots = |X_k|$, and $|X_{k+1} \cup \cdots \cup X_m| \leq \epsilon|V|$. Let R be the union of all $X_i \times X_j$ over the indices $1 \leq i, j \leq k$ with (X_i, X_j) ϵ-regular and $\mu_{X_i,X_j} \geq 2\epsilon$.

Consider the decomposition of f as the sum of Boolean functions f_{b} and f_{s}, where $f_{\mathrm{b}} = \mathbf{1}_R.f$ and $f_s = f - \mathbf{1}_R.f$.

9. Show that f_s has a small norm (as a function of ϵ).

10. Prove the following property for f_{b}. Let n be an integer. An n-cycle $\mathbf{x}$ for f_{b} is an n-tuple $\mathbf{x} = (x_1, x_2, \dots, x_n)$ such that $f_{\mathrm{b}}(x_1, x_2) = f_{\mathrm{b}}(x_2, x_3) = \cdots = f_{\mathrm{b}}(x_n, x_1) = 1$. Show that either f_{b} does not have any n-cycle or it has at least $(2^n - n)\left(\frac{\epsilon(1-\epsilon)}{m}\right)^n |V|^n$ n-cycles.

Hint: *Observe that the number of such n-cycles $\mathbf{x}$ with $\mathbf{x} \in X_{i_1} \times \cdots \times X_{i_n}$ is given by $\sum_{\mathbf{x}\in X_{i_1}\times\cdots\times X_{i_n}} f(x_1, x_2) f(x_2, x_3) \dots f(x_n, x_1)$. Write $f_{|X_i \times X_j} = g_{i,j} + \mu_{X_i, X_j}$ as in 4., and apply 4.*

11. What is the extension of 10. to other patterns?